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Intrawell stochastic resonance of bistable system

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Abstract

In this paper, we describe application of a single-well approximation to a bistable system. Based on this approximation, the relationship between system response speed and steady state variance is obtained. It becomes possible to determine the performance of stochastic resonance (SR) systems by a single measure, the signal-to-noise ratio (SNR) gain. The peaking phenomenon of SNR gain can be found in the single-well-approximated bistable system with the excitation of Gaussian white noise via changing system response speed or sampling period. The mechanisms of some SR phenomena are then discussed, including intrawell SR, parameter-induced SR.

1. Introduction

Stochastic resonance (SR) has been the subject of both theoretical and experimental research in the past two decades [1–30]. The most significant characteristic of SR is that, over a certain range of signal and system parameters, it can cause a transfer of energy from a random process (noise) to a periodic process (signal). SR was initially used to explain certain physical phenomena, such as the earth's climatic change [1–3]. Recently, it has been gaining increasing interest as a potential tool of signal processing [10–18]. But until now, there is not a universal conclusion of the mechanism of SR, which can be used to explain all the phenomena of SR in different systems or conditions, even in bistable systems.

Originally, SR was thought to occur only in systems with bi- or multi-stable potentials. The "classical" description of the phenomenon of SR is that of a particle in a dual-well potential, which is excited by a strong noise and a weak periodic signal [10]. More discussions have been

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summarized in a review paper [6]. Subsequently, it was shown that the signal-to-noise ratio (SNR) could be enhanced by noise for underdamped single-well systems [19] and for a special type of monostable systems [20]. Other interesting examples are given in Refs. [21–23]. Recently, Alfonsi et al. showed us the phenomenon of intrawell SR [24], which is caused by adding Lorentz' colored noise.

On the other hand, the majority of the theoretical studies in this area mentioned above have focused on non-linear systems with additive white noise. However, the behavior similar to what is commonly ascribed to SR has been found in a linear system recently [25–27]. These studies suggested that only noise multiplicativity or time correlation can cause the SR in a linear system. In other words, the classical SR will not occur in a linear system excited with additive Gaussian white noise.

In fact, linear systems are a kind of monostable systems; therefore, it is of significant importance to study SR in monostable systems. Furthermore, a bistable system can also evolve to a monostable system, i.e., the range of a single well. With different system parameters and characteristics of input, there will be two kinds of conditions to make the system output standing in the range of a single well. One is that the system bistability has been destroyed by the input signal; namely, there will be only one potential well. In the case, the system parameters are in suprathreshold region. Another is that the output cannot jump from one potential well to another, usually, the noise intensity is not large enough to cause the interwell jump in a period of signal. In the work of parameter-induced SR, it is found that the SR point mainly located in suprathreshold region [29,30].

More generally, the interwell jump only occurs in short time interval compared with the signal period; other parts of signal only vary in a single well. Previously, we suggested that a recovery formula with a curve fitting could be used to diminish waveform distortion of the output from SR system [28]. Based on the recovery formula, it is found that the final phase lag depends mainly on the intrawell phase lag, while the interwell jump becomes a counter action to signal restoration (see Appendix A). In practice, the phase lag cannot be omitted in processing analog signals. Since the system response speed of intrawell motion is much larger than that of interwell, the final phase lag will be quite small after the recovery formula. Intrawell SR becomes a promising way to process an analog signal, especially a multi-frequency analog signal.

In the following, we will focus on a single well of a bistable system. In the beginning of Section 2, an approximation to simplify the work on intrawell SR is introduced. Then, we will discuss some aspects about it, including the variance of stochastic component in system output, the condition satisfied this approximation and the variance of a recovered signal. In Section 3, we provide a measure, considering both the system response speed and the steady state variance, to determine the performance of SR systems. Based on this measure, several topics are discussed, including the relation between parameter-induced SR and classical SR, the mechanism of intrawell SR and the peaking phenomenon of SNR gain caused by changing sampling period.

2. Single-well approximation of bistable system

Consider the Brownian motion of a particle of mass m, moving in a one-dimensional potential $V(x) = -a^*x^2/2 + \mu^*x^4/4, \quad a^*, \mu^* > 0,$ with damping $c\dot{x}$ and excitation

$H^*(t) = h^*(t) + \Gamma^*(t)$

$$m\ddot{x} + c\dot{x} + V'(x) = h^*(t) + \Gamma^*(t). \quad (1)$$

When the mass of particle is very small, the effect of inertia force can be omitted. Take the variable transforms

$$a^*/c = a, \quad \mu^*/c = \mu, \quad h^*(t)/c = h(t), \quad \Gamma^*(t)/c = \Gamma(t),$$

the system is reduced into

$$\dot{x} = C(x) + h(t) + \Gamma(t), \quad (2)$$

where

$$C(x) = ax - \mu x^3, \quad a, \mu > 0,$$

$\Gamma(t)$ is a Gaussian white noise with zero mean $E[\Gamma(t)] = 0$ and autocorrelation

$$\langle \Gamma(t)\Gamma(t') \rangle = 2D\delta(t - t'). \quad (3)$$

Here, the operator $E[\cdot]$ is ensemble average and $\langle \cdot \rangle$ is sample average, $2D$ is the noise intensity. In practice, the input will be sampled. With a sampling period Δt, the variance of $\Gamma(t)$ is

$$\sigma^2 = D[\Gamma(t)] = 2D/\Delta t. \quad (4)$$

The potential function of system (2) is

$$U(x) = -ax^2/2 + \mu x^4/4 - h(t)x, \quad (5)$$

which is modulated by the input signal $h(t)$.

2.1. The intention of studying intrawell SR

If the input is a constant h instead of a time-varying signal, the steady state solution of the output probability density $\rho(x)$ can be easily obtained [29]. The steady state SNR can be defined as

$$SNR = E^2[x]/D[x],$$

where $E[x] = \int_{-\infty}^{\infty} x\rho(x)\,\mathrm{d}x$ and $D[x] = E[x^2] - (E[x])^2$.

However, the input signal $h(t)$ is a function of the time t and we hope that the system response can trace the changing of the input signal. In this case, the input signal can be treated as a constant during the relaxation time of system response, which is the reciprocal of system response speed λ_1 [29].

In practice, we first determine the system response speed λ_1 depended on the maximum input signal frequency f and sampling frequency f_s. The adopted system response speed satisfies

$$f \ll \lambda_1 \ll f_s.$$

Then we obtain the maximum steady state SNR by carefully tuning the system parameters a and μ [29]. Since the system response speed is limited by the interwell jumping, the adopted input is usually varied quite slowly.

Generally, the system response speed of intrawell motion is much larger than that of interwell jumping. The characteristic time of interwell jumping is at the magnitude of $O(\exp(1/D))$, while that of intrawell motion is at the magnitude of $O(-\ln D)$. When $D \ll 1$, the former is much larger than the latter. Intrawell SR can release the limitation on system response speed and allow a higher input frequency.

Moreover, when handling SR by tuning system parameters, we found that the point corresponding to the optimal system output is usually located in suprathreshold region. Namely, SR phenomenon occurs in system with optimal system parameters is usually intrawell SR. So it is important to explore the phenomenon of intrawell SR.

The single-well approximation will simplify the work on system (2) about intrawell motion. Meanwhile, it does not mean that the single-well approximation can only be used when there is only intrawell motion. With the recovery formula, the interwell jumping can be simply replaced by line segments when the transient time of interwell jumping is short enough as compared to the period of signal (see Appendix A).

2.2. The single-well approximation

Now consider the condition that the value of output $x(t)$ is in the range of a single well. Since the output probability is mainly gathering near a stable state (bottom of a potential well), we can cast a Taylor expansion on potential (5)

$$U^{*}(x) \approx U(x_s) + U'(x_s)(x - x_s) + U''(x_s)(x - x_s)^2/2 + o(x - x_s)^2.$$

Here, x_s is the location of the bottom point of corresponding potential well satisfying

$$U'(x) = -ax + \mu x^3 - h(t) = 0, \tag{6}$$

so that $U'(x_s) = 0$. The $U(x_s)$ is a constant, which can be omitted for a potential function. So the system potential function can be approximately treated as

$$U^{*}(x) \approx U''(x_s)(x - x_s)^2/2. \tag{7}$$

Commonly, there will be more than one solution from Eq. (6), here we take a stable one, therefore, $U''(x_s) > 0$.

Correspondingly, system (2) is then reduced into

$$\dot{x} = -U''(x_s)(x - x_s) + \Gamma(t). \tag{8}$$

In the following, we called the system of Eq. (8) single-well-approximated bistable (SWAB) system. The probability density of the system output $\rho(x,t)$ satisfies the following Fokker–Planck equation (FPE)

$$\frac{\partial \rho(x,t)}{\partial t} = -U''(x_s)\frac{\partial}{\partial x}[(x - x_s)\rho(x,t)] + D\frac{\partial^2}{\partial x^2}\rho(x,t). \tag{9}$$

As mentioned in our previous paper [29], the system response speed λ_1 is the inverse value of a eigenvalue of Eq. (9), which dominates the transient behavior. The system response speed is given as

$$\lambda_1 = U''(x_s).$$

2.3. The stochastic component of output

Because the input signal $h(t)$ is varying with time t, x_s and $U''(x_s)$ are not constant. Usually, $h(t)$ varies in a large time-scale such as signal period. If the system response speed is large enough, the signal can be regarded as constant in a short time interval Δt,

$$1/\lambda_1 \ll \Delta t \ll T,$$

here $1/\lambda_1$ is the system response time, T is the signal period.

With a constant input $h(t) = h$, assume the form of the steady state output to be

$$x(t) = x_s + y(t), \tag{10}$$

here $y(t)$ is a stochastic process. Namely, the output is a combination of deterministic component and stochastic component. Determined by Eq. (6), x_s is a constant now.

Substituting Eq. (10) into Eq. (8), then

$$\dot{y}(t) = -U''(x_s)y(t) + \Gamma(t). \tag{11}$$

It can be viewed as a system with input $\Gamma(t)$ and output $y(t)$. The frequency response for this system is

$$H(\omega) = \frac{1}{U''(x_s) + \mathrm{i}\omega}. \tag{12}$$

The relation of the mean value between input and output is

$$E[y(t)] = \mu_y = H(0)\mu_\Gamma, \tag{13}$$

and the relation of the power spectrum between input and output is

$$S_y(\omega) = |H(\omega)|^2 S_\Gamma(\omega). \tag{14}$$

For $\Gamma(t)$ is a zero mean Gaussian white noise, the mean value of $y(t)$ is also zero. In other words, the mean value of output from system (8) is just x_s. From Eq. (3), the power spectrum of $\Gamma(t)$ is

$$S_\Gamma(\omega) = \int_{-\infty}^{\infty} C_\Gamma(\tau)\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau = 2D.$$

Thus the power spectrum of $y(t)$ is

$$S_y(\omega) = \frac{2D}{U''^2(x_s) + \omega^2},$$

and the auto-correlation function of $y(t)$ is

$$C_y(\tau) = \langle y(t)y(t+\tau)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_y(\omega)\mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\omega = \frac{D}{U''(x_s)}\exp[-U''(x_s)\tau].$$

Therefore, the variance of $y(t)$ is

$$D[y(t)] = \langle y(t)y(t)\rangle = C_y(0) = D/U''(x_s). \tag{15}$$

2.4. Discussion of the single-well approximation

In the following, we estimate the error caused by the approximation. Substituting Eq. (10) into system (2)

$$\dot{y}(t) = ay(t) - 3\mu x_s^2 y(t) - 3\mu x_s y^2(t) - \mu y^3(t) + \Gamma(t). \tag{16}$$

If the condition $|y(t)| \ll 1$ is satisfied, we can omit the higher order terms of $y(t)$, and Eq. (16) can be reduced into Eq. (11).

Consider the following form of

$$y(t) = y_0(t) + y_e(t), \tag{17}$$

here, $y_0(t)$ is the result of the SWAB system, and $y_e(t)$ is the deviation caused by this approximation. Assume that $y_0(t)$ and $y_e(t)$ are uncorrelated. Substituting Eq. (17) into Eq. (16)

$$\begin{aligned}\dot{y}_e(t) &= -U''(x_s)y_e(t) - 3\mu x_s y^2(t) - \mu y^3(t) \\ &\approx -U''(x_s)y_e(t) - 3\mu x_s y_0^2(t) - \mu y_0^3(t),\end{aligned} \tag{18}$$

it can be viewed as a system with input $\xi(t) = -3\mu x_s y_0^2(t) - \mu y_0^3(t)$ and output $y_e(t)$. The frequency response for system (18) is the same as Eq. (12). So the mean value of $y_e(t)$ is

$$E[y_e(t)] = E[\xi(t)]/U''(x_s) = -3\mu x_s D/U''^2(x_s),$$

and the mean value of $x(t)$ is not x_s but

$$E[x(t)] = E[x_s + y_0(t) + y_e(t)] = x_s[1 - 3\mu D/U''^2(x_s)].$$

The relative error of mean value is

$$e_m = |E[x_s(t)] - x_s|/x_s = 3\mu D/U''^2(x_s). \tag{19}$$

Consider the variance of $y_e(t)$

$$D[y_e(t)] = \int_{-\infty}^{\infty} \frac{1}{U''^2(x_s) + \omega^2} S_\xi(\omega)\,\mathrm{d}\omega < \frac{1}{U''^2(x_s)} \int_{-\infty}^{\infty} S_\xi(\omega)\,\mathrm{d}\omega = \frac{D[\xi(t)]}{U''^2(x_s)},$$

and

$$D[\xi(t)] = E[(-3\mu x_s y_0^2(t) - \mu y_0^3(t))^2] = 9\mu^2 x_s^2 E[y_0^4(t)] + \mu^2 E[y_0^6(t)] + 6\mu^2 x_s E[y_0^5(t)].$$

When the deviation is small, we assume that $y_0(t)$ satisfies normal distribution function

$$\rho(y_0(t)) = \exp[-y_0^2(t)/2\sigma_0^2]/\sqrt{2\pi}\sigma_0,$$

where $\sigma_0^2 = D[y_0(t)] = D/U''(x_s)$. Then, $D[\xi(t)]$ and $D[y_e(t)]$ can be given as

$$\begin{aligned}D[\xi(t)] &= 27\mu^2 x_s^2 D^2/U''^2(x_s) + 15\mu^2 D^3/U''^3(x_s), \\ D[y_e(t)] &< [27\mu^2 x_s^2 U''(x_s) + 15\mu^2 D]D^2/U''^5(x_s).\end{aligned}$$

Finally, the variance of $y(t)$ is

$$D[y(t)] = D[y_0(t) + y_e(t)] \leqslant D[y_0(t)] + D[y_e(t)] + 2\sqrt{D[y_0(t)]D[y_e(t)]},$$

the deviation of variance is

$$\Delta D \leqslant D[y_e(t)] + 2\sqrt{D[y_0(t)]D[y_e(t)]}.$$

Simply, we consider the case of small input signal amplitude, then

$$x_s \approx \pm\sqrt{a/\mu}, \quad U''(x_s) \approx 2a.$$

Meanwhile $a/\mu = O(1)$, then $D[y_e(t)]/D[y_0(t)] = O(\sigma_0^2)$.

When $\sigma_0 \ll 1$, the relative error of variance is

$$e_\sigma = \Delta D/D[y_0(t)] \leqslant D[y_e(t)]/D[y_0(t)] + 2\sqrt{D[y_e(t)]/D[y_0(t)]} = O(\sigma_0). \tag{20}$$

Meanwhile, the relative error of mean value is $e_m = O(\sigma_0^2)$. Consequently, the condition satisfying the single-well approximation is

$$\sigma_0 = \sqrt{D[y_0(t)]} = \sqrt{D/U''(x_s)} \ll 1. \tag{21}$$

2.5. The variance of recovered result

With the recovery formula (see Appendix A), output (10) becomes

$$\begin{aligned} h_{out} &= -C(x_s + y(t)) = -C(x_s) - C'(x_s)y(t) + O(y(t))^2 \\ &\approx h - C'(x_s)y(t) = h + U''(x_s)y(t). \end{aligned} \tag{22}$$

The variance of recovered result h_{out} is

$$D[h_{out}] = U''^2(x_s)D[y(t)] = U''(x_s)D. \tag{23}$$

Comparing the variance of recovered result with the variance of $\Gamma(t)$, the ratio of the variances is

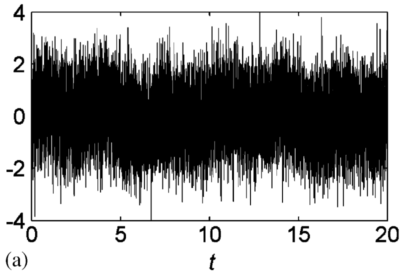
$$\eta = D[h_{out}]/D[\Gamma(t)] = U''(x_s)\Delta t/2. \tag{24}$$

In a previous paper, we presented a simulation example considering an original signal

$$h(t) = 0.18\sin(0.2\pi t) + 0.18\sin(0.6\pi t),$$

combining with a Gaussian white noise of $\sigma^2 = 1$ as the input, and a sampling period $\Delta t = 0.001$. Here, we select the system parameters as $a = \mu = 5$ so that the output is in the range of a single well. The average value of $U''(x_s)$ is about 10.6, and $\eta = U''(x_s)\Delta t/2 \approx 0.0053$. The variance of recovered result is only 0.53 percent of that of original signal.

Figs. 1a and b are the results of the simulation mentioned above. The variance of recovered result (Fig. 1b) is much smaller than that of original signal (Fig. 1a).



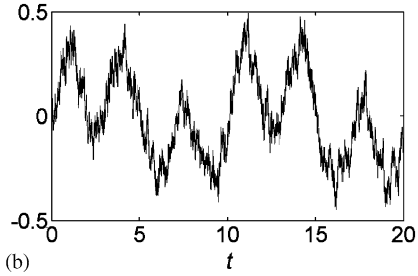


Fig. 1. (a) original input (signal and noise). (b) Recovered result.

3. The peaking phenomenon of SNR gain

3.1. A measure of system performance

Only the steady state solution has been considered in Section 2. We now consider the evolvement of the deterministic component. Assume that this evolvement is the kind of

$$x(t) = x_s - (x_s - x_0)\exp[-U''(x_s)t]. \tag{25}$$

Namely, the output approaches the steady state solution of deterministic component exponentially. The speed of such approach is the system response speed.

According to the recovery formula, the output (from recovery formula) is

$$h_{out} = h + C'(x_s)(x_s - x_0)\exp[-U''(x_s)t] + \Delta h,$$

where $\Delta h = 3\mu x_s(x_s - x_0)^2 \exp[-2U''(x_s)t] + \mu(x_s - x_0)^3 \exp[-3U''(x_s)t]$. The term Δh approaches to zero faster than the other terms. So that we omit Δh. Meanwhile,

$$h_0 = -C(x_0) = -C(x_s + x_0 - x_s) = -C(x_s) - C'(x_s)(x_0 - x_s) + O(x_0 - x_s)^2,$$

when $(x_s - x_0)$ is small, there will be

$$h - h_0 = -C(x_s) - h_0 \approx -C'(x_s)(x_s - x_0),$$
$$h_{out}(t) \approx h - (h - h_0)\exp[-U''(x_s)t]. \tag{26}$$

As mentioned above, the signal can be regarded as a constant in a short time interval when system response speed is large enough. The performance of output corresponding to every segment of signal is important to the performance of the whole output. So we consider a single segment lasts from time $t = t_0$ to time $t = t_0 + T = t_0 + k\Delta t$; Δt is the sampling period and k is a natural number. The signal amplitude is $h = H$ and the original value of recovered result is $h_0 = 0$. In this case, the recovered result is

$$h_{out}(t) = H(1 - \exp[-U''(x_s)t]), \quad t_0 \leqslant t \leqslant t_0 + T. \tag{27}$$

Define the SNR of input and output to be

$$SNR_{in} = \frac{\int_{t_0}^{t_0+T} H^2\,\mathrm{d}t}{\int_{t_0}^{t_0+T} D[\Gamma(t)]\,\mathrm{d}t} = \frac{H^2\Delta t}{2D}, \tag{28}$$

$$SNR_{out} = \frac{H^2}{DU''(x_s)T}\left\{T - \frac{3}{2U''(x_s)} + \frac{2\exp[-U''(x_s)T]}{U''(x_s)} - \frac{\exp[-2U''(x_s)T]}{2U''(x_s)}\right\}. \tag{29}$$

The gain of input and out SNR is given as

$$gain = \frac{2}{\Delta t U''(x_s)}\left\{1 - \frac{3}{2U''(x_s)T} + \frac{2\exp[-U''(x_s)T]}{U''(x_s)T} - \frac{\exp[-2U''(x_s)T]}{2U''(x_s)T}\right\}. \tag{30}$$

Notice that $T = k\Delta t$ and Eqs. (24), (30) can be transformed into

$$gain = \frac{1}{\eta}\left\{1 - \frac{3}{4k\eta} + \frac{\exp[-2k\eta]}{k\eta} - \frac{\exp[-4k\eta]}{4k\eta}\right\}. \tag{31}$$

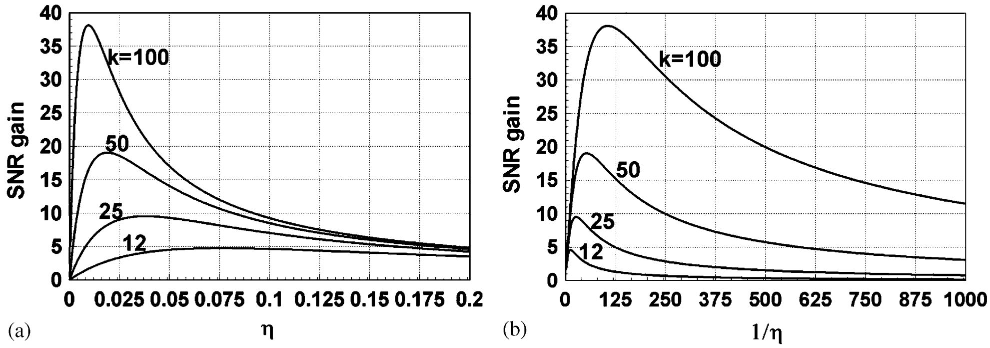


Fig. 2. (a) SNR gain versus η. (b) SNR gain versus $1/\eta$.

The relation of SNR gain and is given in Figs. 2a and b. From the two figures, we can find the peaking phenomenon.

In each of Figs. 2a and b, there are four curves. Each of these curves corresponds to a different k. Here, we present the result with $k = 100, 50, 25$ and 12. With the decreasing of k, the peaking phenomenon is weakening, and even disappears when k is small. Previously, it is proposed that over-sampling is the necessary condition of SR. We can draw the same conclusion from the two figures.

If we fix the sampling period Δt and change $U''(x_s)$, in other words, alter the system parameters to change the system response speed, a peak of SNR gain will present itself with a certain value of $U''(x_s)$, as we proposed previously in Ref. [29]. We named the certain $U''(x_s)$ as critical response speed (CRS).

If we fix the system parameters and change Δt, the peaking phenomenon will also appear. The variance of input noise is $\sigma^2 = 2D/\Delta t = U''(x_s)D/\eta$. In Fig. 2b, with the increasing of $\sigma^2(1/\eta)$, a peaking of the SNR gain occurs.

3.2. Discussion

From the above, changing both the system response speed and the sampling period will induce the behavior similar to SR. But in fact, the SWAB system is equivalent to a linear system, in which the classical SR will not occur when excited with additive Gaussian white noise. The peaking phenomenon here will be ascribed to parameter-induced SR [29]. What is the relation between classical and parameter-induced SR? We will discuss it in the following. Furthermore, the peaking phenomenon caused by changing sampling period will be indicated.

In previous papers [29,30], we proposed that the system response speed and the sampling period are incompatible, therefore, the system response speed and the steady state SNR are not compatible. The phenomenon of SR can be realized by adjusting system parameters. The action of increasing the system response speed will induce larger variance in output while the waveform distortion is decreasing because the system can even follow up the change of noise. Contrarily, decreasing of the system response speed can damp the variety of noise but the change of signal cannot be well followed up.

In a bistable system, the increasing of noise intensity will boost the speed of interwell jumping. Therefore, adding noise is another way to change system response speed. If the system response speed determined by system parameters is less than the CRS, adding noise may cause the system response speed to increase through the CRS to a value beyond the CRS. During this course, the peaking of SNR gain occurs. It is a course that the action of increasing noise intensity enhanced the SNR gain. It is suggested that parameter-induced SR is the base of classical SR.

Meanwhile, the increasing of noise intensity will induce SNR drop of input. Though the seeming SNR gain can be improved, but the SNR of output will not be the best one. Furthermore, if the system response speed determined by system parameters is greater than the CRS, both the SNR of input and SNR gain will drop when the noise intensity is increased. Obviously, the classical SR is sometime restricted and the result via adding noise is worse than that of changing system parameters.

Now, let us talk about the mechanism of intrawell SR. The peaking phenomenon in SWAB system via changing system parameters is the parameter-induced SR of intrawell. According to the analysis above, if any change of system response speed via adding noise can be found, the classical SR will occurs in SWAB system.

In the SWAB system, the system response speed is dominated by intrawell oscillation, while which is dominated by interwell jumping in a bistable system. In this paper, we only discussed the case that the system is excited by Gaussian white noise. The noise intensity affects nothing on the system response speed of SWAB system. The classical SR, which is induced by adding noise, cannot be found.

As mentioned in Section 1, noise multiplicativity and time correlation are necessary conditions for the SR to occur in a linear system. When excited by colored or multiplicative noise, adding noise will effect the system response speed of linear system and the SWAB system. So the classical SR will occur in the SWAB system when excited by colored or multiplicative noise.

Finally, the choosing of a sampling period is a key point in signal processing. Previously, the sampling period is determined by the signal period and the sampling theorem. In work reported in this paper, we found the peaking phenomenon of SNR gain via changing the sampling period. Under certain conditions, tuning sampling period can induce the peaking phenomenon of SNR gain while the system parameters are fixed. It may give some valuable information to signal processing.

Sampling a noise at the time $t = \bar{t}$, the sampling result $\xi(\bar{t})$ is a stochastic variable with zero mean and the variance $\sigma^2 \cdot \xi(\bar{t})$, which is an real variable, will not change with the different sampling period, so the variance σ^2 is independent of the sampling period Δt.

When sampling a Gaussian white noise with different sampling period Δt_1 and Δt_2, the corresponding noise intensities are

$$D_1 = \Delta t_1 \sigma^2/2, \quad D_2 = \Delta t_2 \sigma^2/2,$$

where σ^2 is the real value of the noise variance. From the FPE (9), different noise intensity D will induce different steady state SNR [29]. The changing of sampling period will influence the value of CRS.

Based on Benzi's opinion [1] and our practice, SR systems suppress the noise with continuous spectrum mainly. When the noise is a white noise, the effect is the best. Sampling with smaller sampling period, the bandwidth of noise sample will be wider and the effect will be better.

4. Conclusion and outlook

In this paper, we studied a SWAB system excited with additive Gaussian white noise. The SWAB system is equivalent to a linear system, so the intensity of Gaussian white noise affects nothing on the system response speed. But in such condition, we also demonstrated the peaking phenomenon of SNR gain via tuning system parameters or sampling period. When excited with colored or multiplicative noise, the system response speed of SWAB system will change with the noise intensity. So we suggested that the classical SR would occur in a SWAB system when excited with colored or multiplicative noise.

Though the peaking phenomenon is not the same as previous result of classical SR, it may give some benefits to signal processing via these systems. It is confirmed that the phenomenon of SR may occur via tuning system parameters. On the other hand, the phenomenon that tuning sampling period can induce the peaking phenomenon of SNR gain while the system parameters are fixed may give some valuable information to signal processing. How to determine the optimal sampling period is a valuable topic.

Based on the single-well approximation, we got the relation between system response speed and steady state variance. Then we tentatively provided a measure (the SNR gain in Section 3) considered both system response speed and steady state variance, to determine the performance of SR systems. But the measure we presented in this paper is only a cursory one. It is valuable to construct a particular one considering the waveform of input signal and the phase lag in output signal.

The phase lag is an important factor in analog signal processing. Based on the work of intrawell SR, we can give the method to estimate the phase lag to compensate the output. This work will be presented in another paper.

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Appendix A. Recovery formula and the restriction

Consider the system

$$\dot{x} = C(x) + h(t) + \Gamma(t). \tag{A.1}$$

If $|\dot{x}| \ll |h|$, there will be

$$h \approx -C(x) - \Gamma(t). \tag{A.2}$$

Take the mean value for this equation and notice that $\langle \Gamma(t) \rangle = 0$, then

$$\bar{h}(t) = \langle -C(x) \rangle \approx -C(\tilde{x}). \tag{A.3}$$

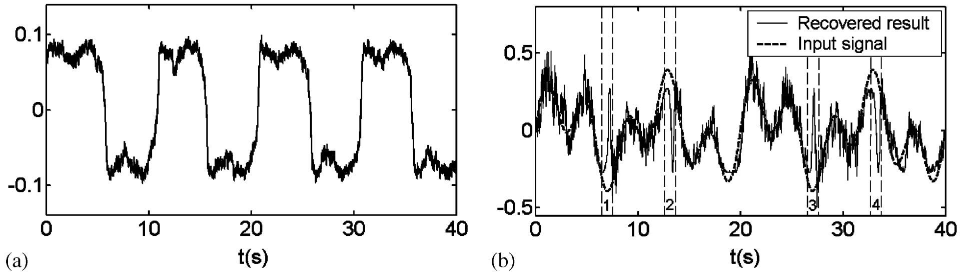

Fig. 3. (a) Output $x(t)$ of a bistable system $\dot{x} = 5x - 250x^3 + h(t) + \Gamma(t)$, where $h(t) = 0.2\sin 0.2\pi t + 0.2\sin 0.5\pi t$, $\Gamma(t)$ is a Gaussian white noise with variance $\sigma^2 = 1$, and the sampling period is $\Delta T = 0.001$ s. (b) The recovered result and original signal.

Here, $\bar{h}(t) = \langle h(t) \rangle$ is the deterministic term, and $\tilde{x}(t) = \langle x(t) \rangle$. With this result, the condition $|\dot{x}| \ll |h|$ becomes

$$|h'(t)| \ll |C'(\tilde{x})h(t)|. \tag{A.4}$$

Obviously, when $h(t)$ is close to zero, condition (A.4) is not satisfied and the output cannot be recovered by Eq. (A.3). A system output is shown in Fig. 3a and the recovered result is compared with the input signal in Fig. 3b. With the comparison between Figs. 3 a and b, it can be found that the error caused by recovery formula occurs accompanying the interwell transition (the jumping between the two potential well). With the recovery formula, the interwell jumping becomes a counteraction to signal restoration. If the transient time is short enough as compared to the period of signal, we only need to correct the recovered result in short time slices in which interwell transition occurs.

The time required for the transition is a statistical value, the expected mean value of which varies with system characteristic time $1/\lambda_1$ (λ_1 is the minimum nonzero eigenvalue for the Fokker–Planck equation of system (2)) [29]. The transient time is diminishing when λ_1 is increasing. Since λ_1 is determined by system parameters and the noise intensity D, we can make the transient time very short by choosing certain system parameters a and μ. For this case, the error part can be simply corrected by linear interpolation.

This information can also be found in another manuscript we submitted recently [28]. Moreover, the intrawell phase lag and other interesting things have been discussed in that paper.

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